UNIFIED FRACTIONAL KINETIC EQUATION AND A FRACTIONAL DIFFUSION EQUATION

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Abstract. In earlier papers Saxena et al. (2002, 2003) derived the solutions of a number of fractional kinetic equations in terms of generalized Mittag-Leffler functions which extended the work of Haubold and Mathai (2000). The object of the present paper is to investigate the solution of a unified form of fractional kinetic equation in which the free term contains any integrable function $f(t)$, which provides the unification and extension of the results given earlier recently by Saxena et al. (2002, 2003). The solution has been developed in terms of the Wright function in a closed form by the method of Laplace transform. Further we derive a closed-form solution of a fractional diffusion equation. The asymptotic expansion of the derived solution with respect to the space variable is also discussed. The results obtained are in a form suitable for numerical computation.

1 Introduction

Fundamental laws of physics are written as equations for the time evolution of a quantity $X(t)$, $dX(t)/dt = -AX(t)$, where this could be Maxwell's equations or Schroedinger's equation (if A is limited to linear operators), or it could be Newton's law of motion or Einstein's equations for geodesics (if A may also be a nonlinear operator). The mathematical solution (for linear operators) is $X(t) = X(0)\text{Exp}\{-At\}$. The initial value of the quantity at $t = 0$ is given by $X(0)$.

The same exponential behavior referred to above arises if $X(t)$ represents the scalar number density of species at time t that do not interact with each other. If one denotes A_p the production rate and A_d the destruction rate, respectively,

the number density $X(t)$ will obey an exponential equation where the coefficient A is equal to the difference, $A_p - A_d$. Subsequently, A_p^{-1} is the average time between production and A_d^{-1} is the average time between destruction. This type of behavior arises frequently in biology, chemistry, and physics (Kaplan and Glass, 1998; Hilfer, 2000; Metzler and Klafter, 2000; Aslam Chaudhry and Zubair, 2002). This paper, in Section 3, considers the fractional generalization of the kinetic equation and derives closed form representation of its solution.

The evolution of the number density $X(x,t)$ relates the first derivative in time of this function to a spatial operator applied to the number density. The initial value of the function at time $t = 0$ is given by $X(x,0)$. In Section 4 of this paper, we give a relationship between the solution of the equation of evolution and the solution of the diffusion equation belonging to its fractional extension (Kaplan and Glass, 1998; Hilfer, 2000; Metzler and Klafter, 2000).

Section 2 summarizes mathematical results concerning solutions of the kinetic and diffusion equations in Sections 3 and 4, respectively, widely distributed in the literature or of very recent origin. These involve the Mittag-Leffler function, Wright function, and H-function, and the application of fractional calculus, Fourier transform, and Laplace transform to them.

Specifically, Haubold and Mathai (1995) discussed a conjecture of a variation of the solar neutrino signal looking at solutions of the standard kinetic equation based on lifetime densities and Poisson arrivals. A closed form representation of the fractional kinetic equation and thermonuclear function is derived by Haubold and Mathai (2000) in terms of certain series which represents the Mittag-Leffler function (Mittag-Leffler, 1903, 1905). In order to extend the work of Haubold and Mathai (2000) three fractional kinetic equations are solved by Saxena et al. (2002) in terms of the generalized Mittag-Leffler functions in a closed form. Further extensions of these results are provided by a recent paper of Saxena et al. (2003). Section 3 and 4 of this paper provide a unification and extension of the aforementioned results on fractional kinetic equations by investigating a closed-form solution of a unified fractional kinetic equation in which the free term contains a function f(t) by the method of Laplace transform. The solution has been obtained in terms of the generalized Wright function in a closed form in Section 3. Also presented is a closed-form solution of a fractional diffusion equation in terms of an H-function. Its asymptotic expansion for large values of the space variable is discussed. The results derived in this paper are in a form suitable for numerical computation.

2 Mathematical Prerequisites

A generalization of the Mittag-Leffler function (Mittag-Leffler, 1903, 1905)

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \quad (\alpha \in C, Re(\alpha) > 0) \tag{1}$$

was introduced by Wiman (1905) in the general form

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad (\alpha, \beta \in C, Re(\alpha) > 0). \tag{2}$$

The main results of these functions are available in the handbook of Erdélyi, Magnus, Oberhettinger and Tricomi (1955, Section 18.1) and the monographs written by Dzherbashyan (1966, 1993). Prabhakar (1971) introduced a generalization of (2) in the form

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(n\alpha + \beta)(n)!}, \quad (\alpha, \beta, \gamma \in C, Re(\alpha) > 0), \tag{3}$$

where $(\gamma)_0 = 1$ is the Pochammer symbol, defined by

$$(\gamma)_0 = 1, (\gamma)_k = \gamma(\gamma + 1) \ldots (\gamma + k - 1) \quad (k = 1, 2, \ldots), \gamma \neq 0. \tag{4}$$

It is an entire function with $\rho = [Re(\alpha)]^{-1}$ (Prabhakar, 1971).
For $\gamma = 1$, this function coincides with (2), while for $\gamma = \beta = 1$ with (1):

$$E_{\alpha,\beta}^{1}(z) = E_{\alpha,\beta}(z), \quad E_{\alpha,1}^{1}(z) = E_{\alpha}(z). \tag{5}$$

We also have

$$\Phi(\beta, \gamma, z) = {}_1F_1(\beta; \gamma; z) = \Gamma(\gamma) E_{1,\gamma}^{\beta}(z) \tag{6}$$

where $\Phi(\beta, \gamma; z)$ is Kummer's confluent hypergeometric function defined in Erdélyi et al. (1953, p.248, eq.1).

The Mellin-Barnes integral representation for this function follows from the integral

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)} \frac{1}{2\pi\omega} \int_{\Omega} \frac{\Gamma(-\xi)\Gamma(\gamma + \xi)(-z)^{\xi}d\xi}{\Gamma(\beta + \xi\alpha)}, \tag{7}$$

where $\omega = (-1)^{1/2}$. The contour Ω is a straight line parallel to the imaginary axis at a distance 'c' from the origin and separating the poles of $\Gamma(-\xi)$ at the points $\xi = \nu$ $(\nu = 0, 1, 2, \ldots)$ from those of $\Gamma(\gamma + \xi)$ at the points $\xi = -\gamma - \nu$ $(\nu = 0, 1, 2, \ldots)$. If we calculate the residues at the poles of $\Gamma(\gamma + \xi)$ at the points $\xi = -\gamma - \nu$ $(\nu = 0, 1, 2, \ldots)$, then it gives the analytic continuation formula of this function in the form

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{(-z)^{-\gamma}}{\Gamma(\gamma)} \sum_{\nu=0}^{\infty} \frac{\Gamma(\gamma + \nu)}{\Gamma[\beta - \alpha(\gamma + \nu)]} \frac{(-z)^{-\nu}}{(\nu)!}, |z| > 1. \tag{8}$$

From (8) it follows that for large z its behavior is given by

$$E_{\alpha,\beta}^{\gamma}(z) \sim O(|z|^{-\gamma}), |z| > 1. \tag{9}$$

3

The H-function is defined by means of a Mellin-Barnes type integral in the following manner (Mathai and Saxena, 1978, p.2):

$$
\begin{aligned}
H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}\left[z\left|{(a_p,A_p) \atop (b_q,B_q)}\right.\right] \\
&= H_{p,q}^{m,n}\left[z\left|{(a_1,A_1),\ldots,(a_p,A_p) \atop (b_1,B_1),\ldots,(b_q,B_q)}\right.\right] = \frac{1}{2\pi\omega}\int_\Omega \Theta(\xi)z^{-\xi}d\xi, \quad (10)
\end{aligned}
$$

where

$$
\Theta(\xi) = \frac{[\prod_{j=1}^m \Gamma(b_j+B_j\xi)][\prod_{j=1}^n \Gamma(1-a_j-A_j\xi)]}{[\prod_{j=m+1}^q \Gamma(1-b_j-B_j\xi)][\prod_{j=n+1}^p \Gamma(a_j+A_j\xi)]}, \quad (11)
$$

$m,n,p,q \in N_0$ with $0 \le n \le p$, $1 \le m \le q$, $A_i, B_j \in R_+, a_i, b_j \in R$ or $C(i=1,\ldots,p; \; j=1,\ldots,q)$ such that

$$
A_i(b_j+k) \neq B_j(a_i-l-1) \;\; (k,l \in N_0; i=1,\ldots,n; j=1,\ldots,m), \quad (12)
$$

where we employ the usual notations: $N_0 = (0,1,2,\ldots); R = (-\infty,\infty), R_+ = (0,\infty)$ and C being the complex number field. Ω is a suitable contour separating the poles of $\Gamma(b_j+sB_j)$ from those of $\Gamma(1-a_j-A_js)$. A detailed and comprehensive account of the H-function is available from Mathai and Saxena (1978).

It follows from (7) that the generalized Mittag -Leffler function $E_{\alpha,\beta}^\gamma(z)$ can be represented in terms of the H-function in the form

$$
E_{\alpha,\beta}^\gamma(z) = \frac{1}{\Gamma(\gamma)}H_{1,2}^{1,1}\left[-z\left|{(1-\gamma,1) \atop (0,1),(1-\beta,\alpha)}\right.\right], (Re(\alpha)>0; \alpha,\beta,\gamma \in C), \quad (13)
$$

and in terms of the Wright function as

$$
E_{\alpha\beta}^\gamma(z) = \frac{1}{\Gamma(\gamma)}\,_1\Psi_1\left[{(\gamma,1) \atop (\beta,\alpha)}|z\right], \quad (14)
$$

where $_1\Psi_1(z)$ is a special case of Wright's generalized hypergeometric function (Wright, 1935, 1940); also see Erdélyi et al. (1953, Section 4.1) where the function is defined by

$$
_p\Psi_q\left[{(a_1,A_1),\ldots,(a_p,A_p) \atop (b_1,B_1),\ldots,(b_q,B_q)}|z\right] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p \Gamma(a_j+nA_j)}{\prod_{j=1}^q \Gamma(b_j+nB_j)}\frac{z^n}{(n)!}, \quad (15)
$$

where $1+\sum_{j=1}^q B_j - \sum_{j=1}^p A_j \ge 0$, (equality only holds for appropriately bounded z). The relation connecting $_p\Psi_q(z)$ and the H-function is given by Mathai and Saxena (1978, p.11, eq. 1.7.8)

$$
_p\Psi_q\left[{(a_1,A_1)\ldots,(a_p,A_p) \atop (b_1,B_1),\ldots,(b_q,B_q)}|z\right] = H_{p,q+1}^{1,p}\left[-z\left|{(1-a_1,A_1),\ldots,(1-a_p,A_p) \atop (0,1),(1-b_1,B_1),\ldots,(1-b_q,B_q)}\right.\right] \quad (16)
$$

It is interesting to observe that for $\gamma = 1$, (13) and (14) give rise to the following results for the generalized Mittag-Leffler function

$$E_{\alpha,\beta}(z) \quad = \quad {}_1\Psi_1 \left[{}^{(1,1)}_{(\beta,\alpha)} | z \right] \tag{17}$$

$$= \quad H^{1,1}_{1,2} \left[-z |^{(0,1)}_{(0,1),(1-\beta,\alpha)} \right] \tag{18}$$

where $Re(\alpha) > 0, \alpha, \beta, \in C$.

If we further take $\beta = 1$ in (17) and (18) we find that

$$E_{\alpha}(z) \quad = \quad {}_1\Psi_1 \left[{}^{(1,1}_{(1,\alpha)} | z \right] \tag{19}$$

$$= \quad H^{1,1}_{1,2} \left[-z \left| {}^{(0,1)}_{(0,1),(0,\alpha)} \right. \right] , \tag{20}$$

where $Re(\alpha) > 0, \alpha \in C$. It is shown by Kilbas et al. (2003) that

$$\int_0^x \quad t^{\nu-1} \quad (x-t)^{\mu-1} E^{\sigma}_{\rho,\nu}(\omega t^\rho) E^{\gamma}_{\rho,\mu}(\omega [x-t]^\rho) dt \tag{21}$$

$$= \quad x^{\nu+\mu-1} E^{\gamma+\sigma}_{\rho,\mu,+\nu}(\omega x^\rho),$$

where $\rho, \mu, \gamma, \sigma, \omega \in C$, $Re(\nu) > 0, Re(\mu) > 0$, which is a generalization of the well-known result of Erdélyi et al.(1953, p.271, eq. 6.10.15)

$$\int_0^x \quad t^{\nu-1} \quad (x-t)^{\mu-1} \Phi(\sigma, \nu, \omega t) \Phi(\gamma, \mu, \omega [x-t]) dt \tag{22}$$

$$= \quad B(\nu, \mu) x^{\nu+\mu-1} \Phi(\gamma + \sigma, \mu + \nu; \omega x),$$

where $Re(\mu) > 0, Re(\nu) > 0$., The Laplace transform of the H-function in terms of another H-function is given by Prudnikov et al. (1989, p.355, eq. 2.25.3)

$$L\left\{ t^{\rho-1} H^{m,n}_{p,q} \left[zt^\sigma \left| {}^{(a_p,A_p)}_{(b_q,B_q)} \right. \right] \right\} = s^{-\rho} H^{m,n+1}_{p+1,q} \left[zs^{-\sigma} \left| {}^{(1-\rho,\sigma),(a_p,A_p)}_{(b_q,B_q)} \right. \right], \tag{23}$$

where $\sigma > 0, Re(s) > 0, Re[\rho + \sigma \min_{1\leq j \leq m}(\frac{b_j}{B_j})] > 0, |arg z| < [\pi/2]\Theta, \Theta > 0$;

$$\Theta = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j. \tag{24}$$

From (23) it can be easily seen that

$$L^{-1} \left\{ s^{-\rho} H^{m,n}_{p,q} \left[zs^\sigma |^{(a_p,A_p)}_{(b_q,B_q)} \right] \right\} = t^{\rho-1} H^{m,n}_{p+1,q} \left[zt^{-\sigma} |^{(a_p,a_p)(\rho,\sigma)}_{(b_q,B_q)} \right], \tag{25}$$

where $\sigma > 0$, $Re(s) > 0$, $Re\left[\rho + \sigma \max_{1\leq j \leq n} \left\{ \frac{1-a_j}{A_j} \right\} \right] > 0$, $|arg z| < \frac{1}{2}\pi\Theta_1, \Theta_1 > 0$, where $\Theta_1 = \Theta - \sigma$, Θ is defined in (24).

5

From Prudnikov et al. (1989, p.355, eq. 2.25.3.2) and Mathai and Saxena (1978, p.49), it follows that the cosine transform of the H-function is given by

$$\int_0^\infty t^{\rho-1} \quad cos(kt) H_{p,q}^{m,n} \left[at^\mu \left| {}^{(a_p,A_p)}_{(b_q,B_q)} \right. \right] dt \tag{26}$$

$$= \quad (\pi/k^\rho) H_{q+1,p+2}^{n+1,m} \left[\frac{k^\mu}{a} \left| {}^{(1-b_q,B_q),(\frac{1}{2}+\frac{\rho}{2},\frac{\mu}{2})}_{(\rho,\mu),(1-a_p,A_p,),(\frac{1}{2}+\frac{\rho}{2},\frac{\mu}{2})} \right. \right],$$

where $Re[\rho + \mu \quad \min_{1\leq j\leq m}(\frac{b_j}{B_j})] > 0, \quad \mu > 0, \quad Re[\rho + \mu \quad \max_{1\leq j\leq n}(\frac{a_j-1}{A_j})] < 1, \quad |arg\ a| < (\pi\theta/2), \Theta > 0, \Theta$ is defined in (24).

The Riemann-Liouville fractional integral of order $\nu \in C$ is defined by Miller and Ross (1993, p.45; see also Srivastava and Saxena, 2001)

$$_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u) du, \tag{27}$$

where $Re(\nu) > 0$. Following Samko et al. (1993, p.37) we define the fractional derivative for $\alpha > 0$ in the form

$$_0D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u)du}{(t-u)^{\alpha-n+1}}, \quad (n = [Re(\alpha)] + 1), \tag{28}$$

where $[Re(\alpha)]$ means the integral part of $Re(\alpha)$.
In particular, if $0 < \alpha < 1$,

$$_0D_t^\alpha f(t) = \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(u)du}{(t-u)^\alpha} \tag{29}$$

and if $\alpha = n \in N = (1, 2, \ldots)$, then

$$_0D_t^n f(t) = D^n f(t)(D = d/dt) \tag{30}$$

is the usual derivative of order n.
From Erdélyi et al. (1954, p.182) we have

$$L\left\{ {}_0D_t^{-\nu} f(t) \right\} = s^{-\nu} F(s), \tag{31}$$

where

$$F(s) = L\left\{ f(t); s \right\} = \int_0^\infty e^{-st} f(t) dt, \tag{32}$$

where $Re(s) > 0$.
The Laplace transform of the fractional derivative is given by Oldham and Spanier (1974, p.134, eq. 8.1.3; see also Srivastava and Saxena, 2001):

$$L[_0D_t^\alpha f(t)] = s^\alpha F(s) - \sum_{k=1}^n s^{k-1} {}_0D_t^{\alpha-k} f(t)|_{t=0}. \tag{33}$$

In certain boundary-value problems arising in the theory of viscoelasticity and in the hereditary solid mechanics, Caputo (1969) introduced the following fractional derivative of order $\alpha > 0$, defined by

$$
\begin{aligned}
D_t^\alpha f(t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)d\tau}{(t-\tau)^{\alpha+1-m}} (m-1 < \alpha \le m), Re(\alpha) > 0, m \in N \\
&= \frac{d^m}{dt^m} f(t), \text{if } \alpha = m
\end{aligned}
\tag{34}
$$

Caputo (1969) has also shown that

$$
L\{D_t^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^k(0+), (m-1 < \alpha \le m).
\tag{35}
$$

The formula (35) is very useful in the solution of differintegral equations governing certain physical problems.

3 Unified Fractional Kinetic Equation

If we integrate the standard kinetic equation, we obtain

$$
N_i(t) - N_0 = -c_i \, _0D_t^{-1} \, N_i(t), \quad (c_i > 0),
\tag{36}
$$

where $_0D_t^{-1}$ is the standard Riemann integral operator. In the paper of Haubold and Mathai (2000) the number density of the species i, $N_i = N_i(t)$, is a function of time and $N_i(t=0) = N_0$ is the number density of species i at time $t = 0$. If the index i is dropped in (36), then the solution of the fractional kinetic equation

$$
N(t) - N_0 = -c^\nu D_t^{-\nu} N(t),
\tag{37}
$$

is obtained in a series form

$$
N(t) = N_0 \sum_{k=0}^\infty \frac{(-c^k t^k)^\nu}{\Gamma(k\nu+1)}.
\tag{38}
$$

Applying the definition (1), the above equation can be rewritten in a compact form in terms of the Mittag-Leffler function

$$
N(t) = N_0 E_\nu[-(ct)^\nu], \nu > 0.
\tag{39}
$$

Theorem 1. If $c > 0, \nu > 0$, then for the solution of the integral equation

$$
N(t) - N_0 f(t) = -c^\nu \, _0D_t^{-\nu} N(t),
\tag{40}
$$

where $f(t)$ is any integrable function on the finite interval $[0, b]$, there holds the formula

$$
N(t) = cN_0 \int_0^t H_{1,2}^{1,1} \left[c^\nu (t-\tau)^\nu \left|_{(-\frac{1}{\nu},1),(0,\nu)}^{(-\frac{1}{\nu},1)} \right. \right] f(\tau)d\tau,
\tag{41}
$$

where $H_{1,2}^{1,1}(.)$ is the H-function defined by (10).

Proof: Applying the Laplace transform to equation (40) and using (31) gives

$$N^*(s) = L[N(t); s] = N_0 \frac{F(s)}{1 + (\frac{c}{s})^\nu}. \tag{42}$$

where $F(s)$ is the Laplace transform of $f(t)$;

Since (Mathai and Saxena, 1978, p. 152)

$$\frac{s^\nu}{s^\nu + c^\nu} = H_{1,1}^{1,1}\left[(\frac{s}{c})^\nu |_{(1.1)}^{(1,1)}\right], \tag{43}$$

then using (25), we obtain

$$L^{-1}\left[H_{1,1}^{1,1}\left[(\frac{s}{c})^\nu |_{(1,1)}^{(1,1)}\right]\right] = t^{-1}H_{2,1}^{1,1}[(ct)^{-\nu}|_{(1,1)}^{(1,1),(0,\nu)}]. \tag{44}$$

By virtue of the following property of the H-function (Mathai and Saxena, 1978, p.4, eq. 1.2.2)

$$H_{p,q}^{m,n}\left[\frac{1}{x}|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{q,p}^{n,m}\left[x|_{(1-a_p,A_p)}^{(1-b_q,B_q)}\right], \tag{45}$$

eq. (44) becomes

$$L^{-1}\left[H_{1,1}^{1,1}\left[(\frac{s}{c})^\nu |_{(1,1)}^{(1,1)}\right]\right] = t^{-1}H_{1,2}^{1,1}\left[(ct)^\nu |_{(0,1),(1,\nu)}^{(0,1)}\right] \tag{46}$$

$$= cH_{1.2}^{1.1}\left[(ct)^\nu \left|_{(-\frac{1}{\nu},1),(0,\nu)}^{(-\frac{1}{\nu},1)}\right.\right]. \tag{47}$$

Eq. (47) follows from (46) if we use the formula (63).

Taking the inverse Laplace transform of (42) by using (47) and applying the convolution theorem of the Laplace transform we arrive at the desired result (41).

If we set $f(t) = t^{\rho-1}$, we obtain the following result established by Saxena et al. (2002, p.283, eq. 15)

Corollary 1.1. If $\nu > 0, \rho > 0, c > 0$, then for the solution of the integral equation

$$N(t) - N_0 t^{\rho-1} = -c^\nu D_t^{-\nu} N(t), \tag{48}$$

there holds the formula

$$N(t) = N_0 \Gamma(\rho) t^{\rho-1} E_{\nu,\rho}[-(ct)^\nu]. \tag{49}$$

Eq. (49) can be established by expressing the H-function, occurring in (41) in terms of an equivalent series by means of the computable representation of the H-function (Mathai and Saxena, 1978, p.71, eq. 3.7.1.) and integrating term by term by means of the beta function formula.

On the other hand, if we take $f(t) = t^{\rho-1} E_{\nu,\mu}^\gamma[-(ct)^\nu]$, then another result recently given by Saxena et al. (2003) is obtained:

8

Corollary 1.2. If $c > 0, \quad \nu > 0, \quad \rho > 0$, then for the solution of the integral equation

$$N(t) - N_0 t^{\mu-1} E_{\nu,\mu}^{\gamma}[-(ct)^{\nu}] = -c^{\nu} \,_0 D_t^{-\nu} N(t), \tag{50}$$

there holds the formula

$$N(t) = N_0 t^{\mu-1} E_{\nu,\mu}^{\gamma+1}[-(ct)^{\nu}]. \tag{51}$$

Eq. (51) can be proved with the help of the results (13) and (21).

4   A Fractional Diffusion Equation

In the following we derive the solution of the fractional diffusion equation using (52). The result is obtained in the form of the following

Theorem 2. Consider the fractional diffusion equation (Metzler and Klafter, 2000; Jorgenson and Lang, 2001)

$$_0 D_t^{\nu} N(x,t) - \frac{t^{-\nu}}{\Gamma(1-\nu)} \delta(x) = c^{\nu} \frac{\partial^2}{\partial x^2} N(x,t), \tag{52}$$

with the initial condition

$$_0 D_t^{\nu-k} N(x,t)|_{t=0} = 0, \quad (k=1,\dots,n), \tag{53}$$

where $n = [Re(\nu)]+1$, c^{ν} is a diffusion constant and $\delta(x)$ is Dirac's delta function. Then for the solution of (52) there exists the formula

$$N(x,t) = \frac{1}{(4\pi c^{\nu} t^{\nu})^{1/2}} H_{1,2}^{2,0} \left[\frac{|x|^2}{4 c^{\nu} t^{\nu}} \Big|_{(0,1),(1/2,1)}^{(1-\frac{\nu}{2},\nu)} \right] \tag{54}$$

Proof. In order to derive the solution of (52), we introduce the Laplace-Fourier transform in the form

$$N^*(k,s) = \int_0^{\infty} \int_{-\infty}^{\infty} e^{-st+ikx} N(x,t) dx dt. \tag{55}$$

Applying the Fourier transform with respect to the space variable x and Laplace transform with respect to the time variable t and using (53), we find that

$$s^{\nu} N^*(k,s) - s^{\nu-1} = -c^{\nu} k^2 N^*(k,s). \tag{56}$$

Solving for $N^*(k,s)$ gives

$$N^*(k,s) = \frac{s^{\nu-1}}{s^{\nu} + c^{\nu} k^2}. \tag{57}$$

9

To invert equation (57), it is convenient to first invert the Laplace transform and then the Fourier transform . Inverting the Laplace transform , we obtain

$$N(k,t) = E_\nu(-c^\nu k^2 t^\nu), \tag{58}$$

which can be expressed in terms of the H-function by using (18) as

$$N(k,t) = H_{1,2}^{1,1}\left[c^\nu k^2 t^\nu \Big|_{(0,1),(0,\nu)}^{(0,1)}\right]. \tag{59}$$

Using the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(k)dk = \frac{1}{\pi}\int_0^\infty f(k)cos(kx)dk, \tag{60}$$

and (26) to invert the Fourier transform, we see that

$$
\begin{aligned}
N(x,t) &= \frac{1}{\pi}\int_0^\infty cos(kx) H_{1,2}^{1,1}\left[c^\nu k^2 t^\nu \Big|_{(0,1),(0,\nu)}^{(0,1)}\right]dk \\
&= \frac{1}{|x|} H_{3,3}^{2,1}\left[\frac{|x|^2}{c^\nu t^\nu}\Big|_{(1,2),(1,1),(1,1)}^{(1,1),(1,\nu),(1,1)}\right].
\end{aligned}
\tag{61}
$$

Applying a result of Mathai and Saxena (1978, p.4, eq. 1.2.1) the above expression becomes

$$N(x,t) = \frac{1}{|x|} H_{2,2}^{2,0}\left[\frac{|x|^2}{c^\nu t^\nu}\Big|_{(1,2),(1,1)}^{(1,\nu),(1,1)}\right]. \tag{62}$$

If we employ the formula (Mathai and Saxena, 1978,p. 4, eq. 1.2.4):

$$x^\sigma H_{p,q}^{m,n}\left[x\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{p,q}^{m,n}\left[x\Big|_{(b_q+\sigma B_q,B_q)}^{(a_p+\sigma A_p,A_p)}\right]. \tag{63}$$

Eq. (62) reduces to

$$N(x,t) = \frac{1}{(c^\nu t^\nu)^{1/2}} H_{2,2}^{2,0}\left[\frac{|x|^2}{c^\nu t^\nu}\Big|_{(0,2),(1/2,1)}^{(1-\frac{\nu}{2},\nu),(1/2,1)}\right]. \tag{64}$$

In view of the identity in Mathai and Saxena (1978, eq. 1.2.1), it yields

$$N(x,t) = \frac{1}{(c^\nu t^\nu)^{1/2}} H_{1,1}^{1,0}\left[\frac{|x|^2}{c^\nu t^\nu}\Big|_{(0,2)}^{(1-\frac{\nu}{2},\nu)}\right]. \tag{65}$$

Using the definition of the H-function (10), it is seen that

$$N(x,t) = \frac{1}{2\pi\omega(c^\nu t^\nu)^{1/2}}\int_\Omega \frac{\Gamma(2\xi)}{\Gamma[1-\frac{\nu}{2}+\nu\xi]}\left[\frac{|x|^2}{c^\nu t^\nu}\right]^{-\xi} d\xi. \tag{66}$$

10

Applying the well-known duplication formula for the gamma function and interpreting the result thus obtained in terms of the H-function, we obtain the solution as

$$N(x,t) = \frac{1}{\sqrt{4\pi c^\nu t^\nu}} H^{2,0}_{1,2} \left[\frac{|x|^2}{4c^\nu t^\nu} \Big|^{(1-\frac{\nu}{2},\nu)}_{(0,1),(1/2,1)} \right].$$ (67)

Finally the application of the result (Mathai and Saxena (1978, p.10, eq. 1.6.3) gives the asymptotic estimate

$$N(x,t) \sim O\left\{ \left[|x|^{\frac{\nu}{2-\nu}}\right] \left[\exp\left\{ -\frac{(2-\nu)(|x|^2\nu^\nu)^{\frac{1}{2-\nu}}}{(4c^\nu t^\nu)^{\frac{1}{2-\nu}}} \right\}\right] \right\} \quad (0 < \nu < 2).$$ (68)

Acknowledgment

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